MATH 4030 Differential Geometry
Problem Set 1
due 22/9/2017 (Fri) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use I, J to denote connected open intervals in $\mathbb{R}$.

1. Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Prove that $L_a^b(\alpha) = L_a^b(\phi \circ \alpha)$. That is, rigid motions preserve the length of curves.

2. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $[a, b] \subset I$. Prove that

$$|\alpha(a) - \alpha(b)| \leq L_a^b(\alpha).$$

In other words, straight lines are the shortest curves joining two given points. *Hint: Use Cauchy-Schwarz inequality.*

3. Let $\alpha : I \to \mathbb{R}^3$ br a curve which does not pass through the origin (i.e. $\alpha(t) \neq 0$ for all $t \in I$). If $\alpha(t_0)$ is the point on the trace of α which is closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

4. Let $\phi : J \to I$ be a diffeomorphism between two open intervals $I, J \subset \mathbb{R}$ and let $\alpha : I \to \mathbb{R}^3$ be a curve. Given $[a, b] \subset J$ with $\phi([a, b]) = [c, d]$, prove that $L_a^b(\alpha \circ \phi) = L_c^d(\alpha)$.

5. Consider the *logarithmic spiral* $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (ae^{bt} \cos t, ae^{bt} \sin t)$$

with $a > 0$, $b < 0$. Compute the arc length function $S : \mathbb{R} \to \mathbb{R}$ from $t_0 \in \mathbb{R}$. Reparametrize this curve by arc length and study its trace.

6. Let $\alpha : I \to \mathbb{R}^2$ be a regular curve (not necessarily parametrized by arc length). Prove that

$$k_\alpha(t) = \frac{1}{|\alpha'(t)|^3} \det(\alpha'(t), \alpha''(t)).$$

7. Let $\alpha : I \to \mathbb{R}^2$ be a plane curve with $[a, b] \subset I$. Suppose $P = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$ is a partition of $[a, b]$. Define

$$L_a^b(\alpha, P) = \sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})|$$

to be the length of the polygonal line. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| L_a^b(\alpha, P) - \int_a^b |\alpha'(t)| \, dt \right| < \epsilon$$

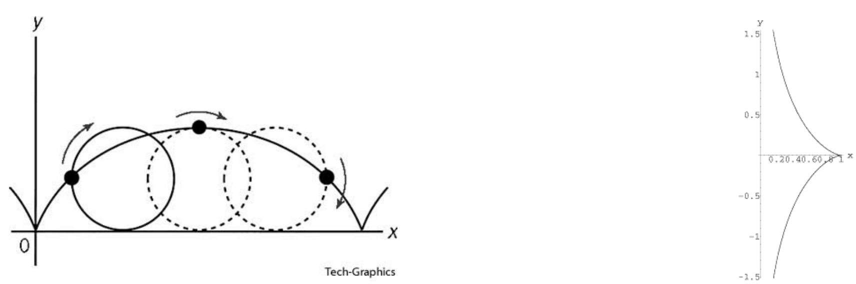
whenever $|P| := \max_{1 \le i \le n} |t_i - t_{i-1}| < \delta$. *Hint: use mean value theorems.* Moreover, show that

$$L_a^b(\alpha) = \sup\{L_a^b(\alpha, P) \mid P \text{ is a partition of } [a, b]\}.$$

8. Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2}),$$

where t is the angle the y-axis make with the vector $\alpha'(t)$. The trace of α is called the *tractrix* (see Figure 1(b)). Show that α is regular except at $t = \pi/2$. Moreover, prove that the length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1.



(a) The cycloid (b) The tractrix

Figure 1

Suggested Exercises

(no need to hand in)

1. Find a curve $\alpha : I \to \mathbb{R}^2$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.
2. Show that if $\alpha : I \to \mathbb{R}^3$ is a curve with $\alpha''(t) = 0$ for all $t \in I$. Show that $\alpha(t)$ is a straight line segment with constant velocity (i.e. $\alpha'(t) =$ a constant vector for all $t \in I$).
3. Let $\alpha : I \to \mathbb{R}^3$ be a curve and $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\langle \alpha'(t), v \rangle = 0$ for all $t \in I$ and that $\langle \alpha(0), v \rangle = 0$. Prove that $\langle \alpha(t), v \rangle = 0$ for all $t \in I$. What does it mean geometrically?

4. Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be the *folium of Descartes* given by

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right).$$

Prove that α is injective but not a homeomorphism onto its image.

5. Consider the *catenary* given by $\alpha : \mathbb{R} \to \mathbb{R}^2$ where $\alpha(t) = (t, \cosh t)$, compute the curvature $k_\alpha(t)$.

6. Let $\alpha : T \to \mathbb{R}^2$ be a curve p.b.a.l. and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion. Show that $\beta = \varphi \circ \alpha : I \to \mathbb{R}^2$ is a curve p.b.a.l. and that

$$k_\beta(s) = \begin{cases} k_\alpha(s) & \text{if } \varphi \text{ is orientation-preserving} \\ -k_\alpha(s) & \text{if } \varphi \text{ is orientation-reversing} \end{cases}$$

7. Let $\alpha : I = (-a, a) \to \mathbb{R}^2$ be a curve p.b.a.l. for some $a > 0$. Suppose that $k_\alpha(s) = k_\alpha(-s)$ for each $s \in (-a, a)$. Prove that the trace of α is symmetric relative to the normal line of α at $s = 0$.

8. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\langle \alpha(t), \alpha'(t) \rangle = 0$ for all $t \in I$.

9. A circular disk of radius 1 in the xy-plane rolls without slipping along the x-axis. The figure described by a point of the circumference of the disk is called a *cycloid* (see Figure 1(a)). Find a curve $\alpha : I \to \mathbb{R}^2$ whose trace is the cycloid. Determine the value of $t \in I$ at which α is not regular. Moreover, compute the arc length of the cycloid corresponding to a complete rotation of the disk.

10. Let $\alpha : I \to \mathbb{R}^2$ be a regular curve such that all the normal lines pass through a fixed point $p \in \mathbb{R}^2$. Prove that the trace $\alpha(I)$ is contained in a circle of some radius $r > 0$ centered at p.

11. Let $\alpha : I \to \mathbb{R}^2$ be a regular curve such that all the tangent lines pass through a fixed point $p \in \mathbb{R}^2$. Prove that the trace $\alpha(I)$ is contained in a straight line passing through p. What if α is not regular?